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The Central Limit Theorem for the Spectrum of the Random One-Dimensional Schrödinger Operator

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Let $H_L = -d^2/dt^2 + q(t, \omega)$ be an one-dimensional random Schrödinger operator in $\mathfrak{L}^2(-L, L)$ with the classical boundary conditions. The random potential $q(t, \omega)$ has a form $q(t, \omega) = F(x_t)$, where x_t is a Brownian motion on the Euclidean ν-dimensional torus, $F: S^\nu \to R^1$ is a smooth function with the nondegenerated critical points, $\min_{S^\nu} F = 0$. Let $N_L(\lambda) = \sum_{\lambda_i(L) \leqslant \lambda} 1(\lambda_i(L, \omega)$ are the eigenvalues of H_L) be a spectral distribution function in the "volume" $[-L, L]$ and $N(\lambda) = \lim_{L\to\infty}(1/2L)N_L(\lambda)$ be a corresponding limit distribution function.

Theorem 1. If $L \to \infty$ then the normalized difference $N_L^*(\lambda) = [N_L(\lambda) - 2L \cdot N(\lambda)]/\sqrt{2L}$ tends (in the sense of Levi–Prokhorov) to the limit Gaussian process $N^*(\lambda)$; $N^*(\lambda) \equiv 0$, $\lambda \leqslant 0$, and $N^*(\lambda)$ has nondegenerated finite-dimensional distributions on the spectrum (i.e., $\lambda > 0$).

Theorem 2. The limit process $N^*(\lambda)$ is a continuous process with the locally independent increments.

KEY WORDS: Random Schrödinger operator; spectrum; limit distribution function; central limit theorem; state density.

INTRODUCTION

In this paper we investigate the random one-dimensional Schrödinger operator

$$H = -\frac{d^2}{dt^2} + q(t, \omega), \qquad t \in R^1, \omega \in \Omega \tag{1}$$

where $(\Omega, \mathfrak{F}, P)$ is the probability space and $q(t, \omega)$ is a stationary (in the narrow sense) random process. One of the basic objects of interest from the point of view of physics and connected with this operator is the limit

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distribution function

$$N(\lambda) = \lim_{L\to\infty} \frac{N_L(\lambda)}{2L} \tag{2}$$

where $N_L(\lambda) = \sum_{\lambda_i(L)\leqslant\lambda} 1$; $\lambda_i(L,\omega)$ are the eigenvalues of the restriction H_L of the operator H to the Hilbert space $\mathfrak{L}^2[-L, L]$ with some classical boundary conditions, for example, Dirichlet conditions

$$y(-L) = y(L) = 0. \tag{3}$$

L. A. Pastur (see Ref. 1) has proved the existence of the nonrandom limit $N(\lambda)$ almost surely. He supposed that the potential $q(t,\omega)$ must be metrically transitive and must almost surely have a lower boundary. L. A. Pastur used substantially the Schturm oscillation theorem and Birghof–Khintchin ergodic theorem.

The formula (2) may be considered a certain law of large numbers for the sequence of spectra of the operator H_L. The natural (and interesting from the point of view of physics) question is one about the estimate of the remainder term in (2). We shall study this problem for the special class of Markov-type potentials $q(t,\omega)$ which were introduced in Ref. 2 and studied in Ref. 3 at great length. (To be precise, we consider a somewhat narrower class of potentials.)

1. In our work the following two results will be established which characterize the Gaussian fluctuation $N_L(\lambda)$ for this special class of the random Schrödinger operators.

Theorem 1. For every compact interval $\Delta \in R^1$, the distributions of the normalized differences

$$N_L^*(\lambda) = \frac{N_L(\lambda) - MN_L(\lambda)}{(2L)^{1/2}}, \qquad \lambda \in \Delta$$

weak-converge in $C(\Delta)$ to the distribution of the continuous Gaussian process $N^*(\lambda)$. The limit process $N^*(\lambda)$ has the following properties: $N^*(\lambda) \equiv 0$ for $\lambda \leqslant 0$ and the finite-dimensional distributions of $N^*(\lambda)$ are nondegenerate if $\lambda > 0$. The correlation function of the process $N^*(\lambda)$ is defined by the formula (11) (see Lemma 3).

Remark 1. Since the process $N_L^*(\lambda)$ is discontinuous the term *weak convergence* wants some specification. As usual we mean the following: the normalized differences $N_L^*(\lambda)$, $\lambda \in \Delta$, can be represented in the form $N_L^*(\lambda) = \tilde{N}_L^*(\lambda) + \epsilon_L(\lambda)$, where $\epsilon_L(\lambda) \to 0$ uniformly in $[0, +\infty)$ (actually $|\epsilon_L(\lambda)| \leqslant C|\sqrt{L}$), the processes $\tilde{N}_L^*(\lambda)$ are continuous and their distributions converge to the distributions of the limit Gaussian process (in Levi–Prokhorov metric on $C(\Delta)$).

Theorem 2. The limit process $N^*(\lambda)$, $\lambda > 0$, has the local Markov property. Moreover, $N^*(\lambda)$ has the locally independent increments. The last means that for every fixed $\lambda > 0$, $n > 0$ the random values

$$\frac{N^*(\lambda + t_1 + t_2) - N^*(\lambda + t_1)}{\{D[N^*(\lambda + t_1 + t_2) - N^*(\lambda + t_1)]\}^{1/2}}, \ldots,$$
$$\frac{N^*(\lambda + t_1 + \cdots + t_{n+1}) - N^*(\lambda + t_1 + \cdots + t_n)}{\{D[N^*(\lambda + t_1 + \cdots + t_{n+1}) - N^*(\lambda + t_1 + \cdots + t_n)]\}^{1/2}}$$

($t_i > 0$, $i = 2, \ldots, n$, $\max_i|t_i| \to 0$, the sign of t_1 is of no importance) are asymptotically independent.

Theorem 2 agrees well with one of the results of Ref. 4 concerning the local Poisson structure of the spectrum of H in $\mathfrak{L}^2(-L, L)$, whenever $L \to \infty$.

2. Let $K = S^\nu$ be the Euclidean ν-dimensional torus and let x_i, $i = 1, \ldots, \nu$, be the natural coordinates on S^ν, $0 \leqslant x_i \leqslant \pi$. The points $0, \pi$ are identified, x_t, $t \in R^1$, is a Brownian motion on S^ν having the stationary (uniform on S^ν) one-dimensional distribution; $F: S^\nu \to R^1$ is a smooth "nonflat" function (the last means that for every $x_0 \in S^\nu$ there exists a number $n = n(x_0)$ such that $d^nF(x_0) \neq 0$). Clearly, the process

$$q(t, \omega) = F(x_t)$$

is the stationary one with the uniformly strong mixing conditions (coefficient of mixing decreases exponentially). Let us also suppose that $\min_{x \in S^\nu} F(x) = 0$. It is proved in Ref. 2 that under this condition the spectrum of H in $\mathfrak{L}^2(R^1)$ coincides with the half-axis $\mathbb{S} = [0, \infty)$. The inner part of the spectrum we shall denote by $\hat{\mathbb{S}}$, that is, $\hat{\mathbb{S}} = (0, \infty)$.

We introduce a phase $\theta_\lambda(s)$ of the equation $Hy = \lambda y$ as usual by the formula

$$\theta_\lambda(s) = \operatorname{arccot} \frac{y}{y'}, \qquad \theta_\lambda(s) \in S^1 \tag{4}$$

As is well known (see Ref. 1),

$$\frac{d\theta_\lambda}{ds} = \cos^2\theta_\lambda(s) + [\lambda - F(x_s)]\sin^2\theta_\lambda(s) \tag{5}$$

and (by the Schturm theorem)

$$N_L(\lambda) = \frac{1}{\pi} \int_{-L}^{L} \{\cos^2\theta_\lambda(s) + [\lambda - F(x_s)]\sin^2\theta_\lambda(s)\}\, ds$$
$$+ R(L), \theta_\lambda(-L) = \frac{\pi}{2} \tag{6}$$

where $|R(L)| \leqslant 1$.

Thus we have reduced the analysis of the behavior of $N_L(\lambda)$ at the infinity to the analysis of the additive functional from the process $(x_t, \theta_\lambda(t))$. In the future we shall also use the properties of the more general process

$$X_n(t) = (x_t, \theta_{\lambda_1}(t), \ldots, \theta_{\lambda_n}(t)), \lambda_1 < \cdots < \lambda_n$$

It is easy to see that the process $X_n(t)$ is the Markov diffusion process (on the compact $K_n = S^\nu \times (S^1)^n$) having the infinitesimal operator

$$A_n = \frac{1}{2}\Delta_x + \sum_{i=1}^{n} \left[\cos^2\theta_i + (\lambda_i - F(x))\sin^2\theta_i\right] \frac{\partial}{\partial\theta_i} \tag{7}$$

and the natural periodic boundary conditions. This operator is degenerate (elliptic–parabolical), hence the problem of the existence and smoothness of transition densities $p(t, (x, \theta_1, \ldots, \theta_n), (x', \theta_1', \ldots, \theta_n'))$ with respect to the Euclidean measure on $S^\nu \times (S^1)^n$ is nontrivial. We shall rely here on the general theory based on Hörmander's ideas of Ref. 5. This theory was first used in a similar situation in Ref. 2.

The following two lemmas generalize the results of Refs. 2 and 3, where they were proved for $n = 1$ and formulated for $n = 2$.

Lemma 1. If $\lambda_1 < \cdots < \lambda_n$, then for $t > 0$ there exists the smooth (in all arguments) transition density $p(t, (x, \theta_1, \ldots, \theta_n), (x', \theta_1', \ldots, \theta_n'))$ of the process $X_n(t)$. This density is the fundamental solution of the equation

$$\frac{\partial p}{\partial t} = A_n p$$

in the cylinder $(0, \infty) \times (S^\nu \times (S^1)^n)$.

Lemma 2. If $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$ (i.e., $\lambda_i \in \hat{S}$, $i = 1, \ldots, n$), then for $t > t_0 = t_0(\lambda_1, \ldots, \lambda_n)$

$$p(t, (\cdot, \cdot, \ldots, \cdot), (\cdot, \cdot, \ldots, \cdot)) > 0$$

Lemma 1 enables us to see that the multidimensional process $(x_t, \theta_{\lambda_1}(t), \ldots, \theta_{\lambda_n}(t))$ satisfies the Doëblin condition for all $\lambda_1 < \cdots < \lambda_n$. According to Lemma 2 this process is uniformly ergodic and connected on $\hat{\mathfrak{S}}$.

Proof of Lemma 1. Let us introduce the vector fields on $S^\nu \times (S^1)^n$:

$$X_i = \frac{\partial}{\partial x_i}, \qquad i = 1, \ldots, \nu$$

$$Y = \sum_{i=1}^{n} \left[\cos^2\theta_i + (\lambda_i - F(x))\sin^2\theta_i\right] \frac{\partial}{\partial\theta_i}$$

By the general theory of Ref. 5, in order to prove Lemma 1 it is sufficient to show that the Lie algebra $\mathfrak{A}$ of vector fields which is generalized by the fields X_i, $i = 1, \ldots, \nu$ and all possible commutators of the form $[\ldots[X_i \ldots [X_k, Y] \ldots Y \ldots]$ (the field Y itself is not included in the number of generators) has the maximum dimension $n + \nu$ at the every point of the manifold $S^\nu \times (S^1)^n$. Let us stress that this algebra is constructed over the ring of the infinitely differentiable functions.

Now we fix a point $(x_0, \theta^0) \in S^\nu \times (S^1)^n$ and consider the commutators of the form $[Y, X_{i_1}, \ldots, X_{i_k}] = Z^{(1)}_{i_1, \ldots, i_k}$. The simple computations show that

$$Z^{(1)}_{i_1, \ldots, i_k} = (-1)^k \frac{\partial^k F(x)}{\partial x_{i_1} \cdots \partial x_{i_k}} \sum_{i=1}^{n} \sin^2\theta_i \frac{\partial}{\partial \theta_i}$$

Because of the "nonflatness" of $F(x)$ there exists a number $k = k(x_0)$ such that

$$\left. \frac{\partial^k F(x)}{\partial x_{i_1} \ldots \partial x_{i_k}} \right|_{x = x_0} \neq 0$$

Since the last derivative is not equal to zero in some neighborhood of (x_0, θ^0), the field $\sum_{i=1}^n \sin^2\theta_i(\partial/\partial\theta_i)$ belongs to $\mathfrak{A}$ (in the fixed neighborhood of the point (x_0, θ^0); all the following discussions will be held in this neighborhood).

Let us construct the commutator $[Y, Z^{(1)}_{i_1, \ldots, i_k}] = Z^{(2)}_{i_1, \ldots, i_k}$. One can see that $Z^{(2)}_{i_1, \ldots, i_k} = \sum_{i=1}^n \sin 2\theta_i(\partial/\partial\theta_i) \in \mathfrak{A}$. Subtracting the field $C \cdot Z^{(1)}$ (C is a suitable coefficient) from the field $Z^{(3)}_{i_1, \ldots, i_k} = [Y, Z^{(2)}_{i_1, \ldots, i_k}] = \sum_{i=1}^n 2[\sin^2\theta_i(\lambda_i - F - 1) - (1 - \sin^2\theta_i)](\partial/\partial\theta_i)$ we have $\hat{Z}^{(3)} = \sum_{i=1}^n \cdot (\lambda_i \sin^2\theta_i - 1)(\partial/\partial\theta_i) \in \mathfrak{A}$. After commutating $[\hat{Z}^{(3)}, Y]$ we note that $\hat{Z}^{(4)} = \sum_{i=1}^n \lambda_i^2 \sin 2\theta_i(\partial/\partial\theta_i) \in \mathfrak{A}$. By induction it is easy to show that for $k \geqslant 2$ $\hat{Z}^{(2k+2)} = \sum_{i=1}^n \lambda_i^k \sin 2\theta_i(\partial/\partial\theta_i) \in \mathfrak{A}$ and $\hat{Z}^{(2k+1)} = \sum_{i=1}^n \cdot [\lambda_i^k \sin^2\theta_i - \lambda_i^{k-1} + P(\lambda_i, F)](\partial/\partial\theta_i)$ where $P(\lambda_i, F)$ is the multinomial of $(k-2)$th degree. Let us suppose now that $\theta^0 = (\theta_1^0, \ldots, \theta_n^0) \in (S^1)^n$ is such that $\sin 2\theta_1^0 \neq 0, \ldots, \sin 2\theta_n^0 \neq 0$. Then (using a property of the Wandermond determinant) we find that the fields $X_1, \ldots, X_\nu, \hat{Z}^{(2)}, \hat{Z}^{(4)}, \ldots, \hat{Z}^{(2n)}$ form a basis of the maximum dimension at $(x_0, \theta^0) \in S^\nu \times (S^1)^n$. If for some i $\sin 2\theta_i^0 = 0$, then we can construct the desirable basis using the fields $\hat{Z}^{(2k+1)}$, $k = 1, \ldots, n$. Lemma 1 is proved. ■

Proof of Lemma 2. One can find such $a_1, \ldots, a_n \in R$, that

(a) $0 < a_1 < \cdots < a_n \leqslant \max_{x \in S^\nu} F(x)$;

(b) $a_1, \ldots, a_n < \lambda_1$;

(c) $\beta_{ij} = (\lambda_i - a_j)^{1/2}$

are rationally independent for all $i, j = 1, \ldots, n$.

Evidently, this may be achieved by various methods.

The system

$$\frac{d^2y_1}{dt^2} = (\lambda_1 - a_i)y_1$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{d^2y_n}{dt^2} = (\lambda_n - a_i)y_n, t \in (t_{i-1}, t_i), t_0 = 0, t_n = T, i = 1, \ldots, n,$$

with the piecewise constant coefficients has an exact solution. We denote the set of phases of these equations at the point T by $(\theta_1(T), \ldots, \theta_n(T))$. We consider the mapping $(t_1, \ldots, t_{n-1}, T) \to (\theta_1(t_1, \ldots, t_{n-1}, T), \ldots, \theta_n(t_1, \ldots, t_{n-1}, T))$. Both the Weil theorem about the irrational winding of the torus and the rational independence of $\beta_{ij} = (\lambda_i - a_j)^{1/2}, i, j = 1, \ldots, n$, yield that the image of the symplex $(0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = T)$ for sufficiently large T coincides with the torus $(S^1)^n$ by such mapping. From this simple fact it follows immediately that for every initial point $(x_0, \theta_1^0, \ldots, \theta_n^0)$ distribution of the process $X_n(t) = (x_t, \theta_{\lambda_1}(t), \ldots, \theta_{\lambda_n}(t))$ at the moment $t = T$ is dense on $S^\nu \times (S^1)^n$. In fact, let us consider some point $(x_1, \theta_1^1, \ldots, \theta_n^1) \in S^\nu \times (S^1)^n$ and some neighborhood $V \ni (x_1, \theta_1^1, \ldots, \theta_n^1)$. We choose $A_1, \ldots, A_n \in S^\nu$ such that $F(A_i) = a_i, i = 1, \ldots, n$, and analyze the following "behavior" of the process $x_t \in S^\nu$ for some sufficiently small numbers $\delta_0, \ldots, \delta_n, \epsilon_1, \ldots, \epsilon_{n+1}$. During the time δ_0 the process x_t moves from x_0 to the ϵ_1 neighborhood of A_1 and remains there from the moment δ_0 till the moment t_1, then during the time which does not exceed δ_1 it moves from the ϵ_1 neighborhood of A_1 to the ϵ_2 neighborhood of A_2. It remains there till the moment $t_2, \ldots,$ at the last but one step [in the interval $(t_{n-1}, t_{n-1} + \delta_{n-1})$] it moves to the ϵ_n neighborhood of A_n, where it stays till the moment $t_n - \delta_n$, and lastly during the time from $t_n - \delta_n$ till $t_n = T$ it goes to the ϵ_{n+1} neighborhood of the point A^1, where it stays till the moment $t_n = T$.

Clearly, this motion has a positive probability. We are left to choose $\delta_0, \ldots, \delta_n, \epsilon_1, \ldots, \epsilon_{n+1}$ sufficiently small and using the above given remark to fix the desirable $t_1, \ldots, t_{n-1}$. Now by the continuous dependence of the solution of the differential equation on the parameters we have that

$$P\big(T, (x_0, \theta_1^0, \ldots, \theta_n^0), V(x_1, \theta_1^1, \ldots, \theta_n^1)\big) > 0 \tag{$*$}$$

We shall show that

$$p(2T, (\cdot, \cdot, \ldots, \cdot), (\cdot, \cdot, \ldots, \cdot)) > 0 \tag{$**$}$$

Indeed, let us suppose that it is not true. Then there are points

$(x_0, \theta_1^0, \ldots, \theta_n^0)$ and $(x_1, \theta_1^1, \ldots, \theta_n^1)$ such that

$$\begin{aligned} &p\big(2T, (x_0, \theta_1^0, \ldots, \theta_n^0), (x_1, \theta_1^1, \ldots, \theta_n^1)\big) \\ &\quad = \int p\big(T, (x_0, \theta_1^0, \ldots, \theta_n^0), (x, \theta_1, \ldots, \theta_n)\big) \\ &\quad \times p\big(T, (x, \theta_1, \ldots, \theta_n), (x_1, \theta_1^1, \ldots, \theta_n^1)\big) \\ &\quad \times dx\, d\theta_1 \ldots d\theta_n = 0 \end{aligned}$$

This yields immediately that $p(T, (x, \theta_1, \ldots, \theta_n), (x_1, \theta_1^1, \ldots, \theta_n^1)) \equiv 0$ (as the function from the arguments $(x, \theta_1, \ldots, \theta_n)$). Applying the Kolmogorov–Chapman equation once more we see that

$$\begin{aligned} &p\big(T + \tau, (x, \theta_1, \ldots, \theta_n), (x_1, \theta_1^1, \ldots, \theta_n^1)\big) \\ &\quad = \int p\big(\tau, (x, \theta_1, \ldots, \theta_n), (\tilde{x}, \tilde{\theta}_1, \ldots, \tilde{\theta}_n)\big) \\ &\quad \times p\big(T, (\tilde{x}, \tilde{\theta}_1, \ldots, \tilde{\theta}_n), (x_1, \theta_1^1, \ldots, \theta_n^1)\big) \\ &\quad \times d\tilde{x}\, d\tilde{\theta}_1 \ldots d\tilde{\theta}_n \equiv 0 \end{aligned}$$

But the last equality can be written down as

$$\begin{aligned} 0 \equiv &\int p\big(T, (x, \theta_1, \ldots, \theta_n), (\tilde{x}, \tilde{\theta}_1, \ldots, \tilde{\theta}_n)\big) \\ &\times p\big(\tau, (\tilde{x}, \tilde{\theta}_1, \ldots, \tilde{\theta}_n), (x_1, \theta_1^1, \ldots, \theta_n^1)\big) d\tilde{x}\, d\tilde{\theta}_1 \ldots d\tilde{\theta}_n \end{aligned}$$

This means that $p(\tau, (\tilde{x}, \tilde{\theta}_1, \ldots, \tilde{\theta}_n), (x_1, \theta_1^1, \ldots, \theta_n^1)) \equiv 0$. But from the definition of the fundamental solution of a differential equation it follows that

$$\begin{aligned} &p\big(\tau, (\tilde{x}, \tilde{\theta}_1, \ldots, \tilde{\theta}_n), (x_1, \theta_1^1, \ldots, \theta_n^1)\big)_{\tau \to 0} \\ &\quad \to \delta\big(\tilde{x}, \tilde{\theta}_1, \ldots, \tilde{\theta}_n), (x_1, \theta_1^1, \ldots, \theta_n^1\big). \end{aligned}$$

We came to a contradiction which proves Lemma 2. ■

Corollary 1. If $0 < \lambda_1 < \cdots < \lambda_n$, then

$$p\big(t, (x, \theta_1, \ldots, \theta_n), (x_1, \theta_1^1, \ldots, \theta_n^1)\big)_{t \to \infty} \to \pi_{\lambda_1, \ldots, \lambda_n}(x_1, \theta_1^1, \ldots, \theta_n^1)$$

with the exponential speed. The limit invariant density is the unique positive solution (within normalization) of the equation $A_n^* \pi_{\lambda_1, \ldots, \lambda_n} = 0$.

Remark 2. One can show, that for any $\lambda_1 < \cdots < \lambda_n$ the process $(x_t, \theta_{\lambda_1}(t), \ldots, \theta_{\lambda_n}(t))$ has the unique ergodic class so that

$$p\big(t, (\cdot, \cdot, \ldots, \cdot), (x, \theta_1, \ldots, \theta_n)\big)_{t \to \infty} \to \pi_{\lambda_1, \ldots, \lambda_n}(x, \theta_1, \ldots, \theta_n)$$

(with the exponential speed). But (see Ref. 2) if at least one of $\lambda_i \leqslant 0$, $i = 1, 2, \ldots, n$, this class is a proper subset of the compact $S^\nu \times (S^1)^n$.

3. The following important limit theorem belongs essentially to S. V. Nagajev (see Ref. 6). Our formulation differs from the Nagajev's theorem in some technical details.

Lemma 3. Let K be a compact, ρ, a dense measure on the Borell σ algebra $\mathfrak{B}(K)$, and let $x_t, t \geqslant 0$, be the Markov homogeneous process having a continuous transition density $p(t, x, y)$ relative to ρ. If there is t_0 such that $p(t_0, x, y) > 0$, then

(a) $p(t, x, y)_{t\to\infty} \to \pi(y)$ (it converges with the exponential speed in the metric of $C(K)$).

(b) If the function $f = (f_1, \ldots, f_n) : K \to R^1$ is measurable and $\int \|f\|^2 d\rho < \infty$, then

$$(\mathrm{b}_1) \quad M_x \int_0^t f(x_s)\, ds = t \int_K f(x)\pi(x)\, d\rho(x) + O(1)$$

$$(\mathrm{b}_2) \quad \operatorname{cov}\left[\int_0^t f_i(x_s)\, ds, \int_0^t f_j(x_s)\, ds\right] = tB_{ij} + O(1)$$

(uniformly in $x \in K$).

(c) If $\det\{B_{ij}\} > 0$, then the distribution of the normalized vector

$$\left\{ \xi_i(t) = \frac{\int_0^t f_i(x_s)\, ds - t\int_K f_i \cdot \pi(x)\, d\rho}{\sqrt{t}}, i = 1, \ldots, n \right\}$$

weak-converge to the nondegenerate Gaussian n-dimensional distribution with the mean equaling to 0 and covariance matrix $B = (B_{ij})_{i,j=1,\ldots,n}$.

(d) If $\det B = 0$, then there exist the constants $c_1, \ldots, c_n$ such that

$$D\left[\sum_{i=1}^n c_i \int_0^t f_i(x_s)\, ds\right] = O(1)$$

when $t \to \infty$.

To use Lemma 3 while examining the process $N_L^*(\lambda) = [N_L(\lambda) - MN_L(\lambda)]/(2L)^{1/2}$ we must effectively compute the second moments of the functionals

$$\int_{-L}^{L} \left[\cos^2\theta_{\lambda_i}(s) + (\lambda_i - F(x_s))\sin^2\theta_{\lambda_i}(s)\right] ds = \varphi_{\lambda_i}(t)$$

One can see that

$$M_{(x,\theta)} \frac{1}{2L} \int_{-L}^{L} \left[\cos^2\theta_\lambda(s) + (\lambda - F(x_s))\sin^2\theta_\lambda(s)\right] ds$$
$$= \int_{S^\nu \times S^1} \left[\cos^2\theta + (\lambda - F(x))\sin^2\theta\right] \pi_\lambda(x, \theta)\, dx\, d\theta \tag{8}$$

We denote $\cos^2\theta + (\lambda - F(x))\sin^2\theta$ by $f_\lambda(x,\theta)$ and $f_\lambda(x,\theta) - \int_{S^\nu \times S^1} f_\lambda(x,\theta) \cdot \pi_\lambda(x,\theta)\,dx\,d\theta$ by $\bar{f}_\lambda(x,\theta)$. Clearly, $\int \bar{f}_\lambda \pi_\lambda\,dx\,d\theta = 0$, hence there exists the unique solution of the equation

$$A_\lambda u_\lambda(x,\theta) = \frac{1}{2}\Delta u_\lambda(x,\theta) + f_\lambda(x,\theta)\frac{\partial u_\lambda(x,\theta)}{\partial\theta} = \bar{f}_\lambda(x,\theta) \tag{9}$$

Applying the Ito formula, it is easy to obtain that

$$\begin{aligned} &u_\lambda(x_t,\theta_\lambda(t)) - u_\lambda(x_0,\theta_\lambda^0) \\ &= \int_0^t \operatorname{grad}_x u_\lambda(x_s,\theta_\lambda(s))\,dw_s + \int_0^t A_\lambda u_\lambda(x_s,\theta_\lambda(s))\,ds \\ &= \int_0^t \operatorname{grad}_x u_\lambda(x_s,\theta_\lambda(s))\,dw_s + \int_0^t \bar{f}(x_s,\theta_\lambda(s))\,ds \end{aligned} \tag{10}$$

where w_s is the ν-dimensional Wiener process on the torus S^ν. We may also hold that w_s is a Wiener process on R^ν, since the function $u_\lambda(x,\theta)$ can be considered as a periodic function on $R^\nu \times R^1$ in each argument (with the period π).

From the formulas (8) and (10) it immediately follows that for μ,λ, $\mu \neq \lambda$,

$$\begin{aligned} R(\mu,\lambda) &= \lim_{L\to\infty}\frac{1}{2L}\operatorname{cov}\left[\int_{-L}^{L} f_\lambda(x_s,\theta_\lambda(s))\,ds, \int_{-L}^{L} f_\mu(x_s,\theta_\mu(s))\,ds\right] \\ &= \lim_{L\to\infty}\frac{1}{2L}\operatorname{cov}\left(\int_{-L}^{L}\operatorname{grad}_x u_\lambda\,dw_s, \int_{-L}^{L}\operatorname{grad}_x u_\mu\,dw_s\right) \\ &= \lim_{L\to\infty}\frac{1}{2L} M\int_{-L}^{L}(\operatorname{grad}_x u_\lambda, \operatorname{grad}_x u_\mu)\,ds \\ &= \int_{S^\nu\times S^1\times S^1}(\operatorname{grad}_x u_\lambda(x,\theta_\lambda), \operatorname{grad}_x u_\mu(x,\theta_\mu)) \\ &\quad \times \pi_{\lambda,\mu}(x,\theta_\lambda,\theta_\mu)\,dx\,d\theta_\lambda\,d\theta_\mu \end{aligned} \tag{11}$$

To obtain this result we have used Corollary 1.

The formula (11) gives us the correlation function of the process $N^*(\lambda)$. But neither the existence of this process nor a kind of the convergence to it have not yet been established.

4. Let us pass on directly to proving Theorem 1. Since for $\lambda \leqslant 0$, $N_L(\lambda) \equiv 0$, then (without losing generality) it is sufficient to carry out all the discussions for the interval Δ of the axis $[0,\infty)$.

The functions $u_\lambda(x,\theta)$ are continuous in (x,θ,λ), hence they are bounded. The transition density $p_\lambda(t,(x,\theta),(x_1,\theta_1))$ converges to the invariant density $\pi_\lambda(x,\theta)$ uniformly in $\lambda \in \Delta$ (except a neighborhood of $\lambda = 0$, but $p_\lambda(t,(x,\theta),(x_1,\theta_1))$ is uniformly bounded, therefore the Döeblin condition holds uniformly in λ).

From these facts it follows that

$$N_L^*(\lambda) = \frac{N_L(\lambda) - MN_L(\lambda)}{(2L)^{1/2}}$$

$$= \left[\frac{1}{\pi} \int_{-L}^{L} (\operatorname{grad} u_\lambda(x_s, \theta_\lambda(s)), dw_s) \right] \Big/ (2L)^{1/2} + \frac{R_1(\lambda, L)}{(2L)^{1/2}}$$

where $R_1(\lambda, L)$ is uniformly bounded in $\lambda \in \Delta$ as $L \to \infty$. This means that to prove the weak convergence of the distributions of $N_L^*(\lambda)$ in sup metric, we must show the weak convergence for the sequence of the continuous processes (and martingals at that)

$$\tilde{N}_L^*(\lambda) = \frac{1}{\pi(2L)^{1/2}} \int_{-L}^{L} (\operatorname{grad} u_\lambda(x_s, \theta_\lambda(s)), dw_s) \tag{12}$$

According to the general Prokhorov's theorem (see Ref. 7) to finish the proof of Theorem 1 we are left to establish two conditions: (a) the convergence of the finite-dimensional distributions of the processes $N_L^*(\lambda), \lambda \in \Delta$, to the Gaussian limit distributions, and (b) a compactness of the distributions of the processes $\tilde{N}_L^*(\lambda)$ in $C(\Delta)$.

Lemma 3 yields the weak convergence of the finite-dimensional distributions of $N_L^*(\lambda), \lambda \in \Delta$ to the corresponding finite-dimensional distributions of the Gaussian process $N^*(\lambda)$ with the null mean and correlation function $R(\lambda, \mu)$ [see formula (11)].

We verify the fact that these finite-dimensional distributions are nondegenerate. To achieve this we use Lemma 3, Part (c).

Let $0 < \lambda_1 < \cdots < \lambda_n$. We consider the function $\tilde{f} = \sum_{i=1}^{n} c_i f_{\lambda_i}(x, \theta_i)$ and the corresponding additive functional

$$\tilde{\varphi}(L) = \int_{-L}^{L} \sum_{i=1}^{n} C_i f_{\lambda_i}(x_s, \theta_{\lambda_i}(s))\, ds$$

Repeating the discussion given in Section 3 we see that

$$\lim_{L\to\infty} \frac{D\tilde{\varphi}(L)}{2L} = \int_{S^\nu \times (S^1)^n} \operatorname{grad}_x^2 \tilde{u}_{\lambda_1, \ldots, \lambda_n}(x, \theta_1, \ldots, \theta_n)\, \pi_{\lambda_1, \ldots, \lambda_n} dx\, d\theta_1 \ldots d\theta_n$$

where $\tilde{u}_{\lambda_1, \ldots, \lambda_n}(x, \theta_1, \ldots, \theta_n)$ is the solution of the equation

$$A_{\lambda_1, \ldots, \lambda_n} \tilde{u} = \sum_{i=1}^{n} c_i \bar{f}_{\lambda_i}(x, \theta_i)$$

Plainly,

$$\tilde{u}_{\lambda_1,\ldots,\lambda_n}(x,\theta_1,\ldots,\theta_n)=\sum_{i=1}^{n}c_i u_{\lambda_i}(x,\theta_i)$$

Since $\pi_{\lambda_1,\ldots,\lambda_n}(x,\theta_1,\ldots,\theta_n)>0$, therefore

$$\sum_{i=1}^{n}c_i \operatorname{grad}_x u_{\lambda_i}(x,\theta_i)\equiv 0$$

whenever $\lim_{L\to\infty}[D\tilde{\varphi}(L)/2L]=0$.

From this fact it follows that

$$\operatorname{grad}_x u_{\lambda_i}(x,\theta_i)=g_i(x)$$

where $g_i, i=1,\ldots,n$, are some vector functions and

$$u_\lambda(x,\theta)=G_\lambda(x)+h_\lambda(\theta)$$

which leads to a contradiction with the equation

$$\frac{1}{2}\Delta u_\lambda+f_\lambda(x,\theta)\frac{\partial u_\lambda}{\partial\theta}=\bar{f}_\lambda(x,\theta)$$

Now let us check an equicontinuity of the family of the processes $\tilde{N}_L^*(\lambda)$, which provides us with the desirable compactness. By the well-known version of the Kolmogorov theorem about the continuity of the random processes it is sufficient to establish that $M_x|\tilde{N}_L^*(\lambda+h)-\tilde{N}_L^*(\lambda)|^4\leqslant \text{const}\cdot h^{1+\alpha}, \alpha>0$ (const and α are independent of L).

The inequality

$$M\left|\int_0^T f(t)\,dw(t)\right|^p\leqslant B_p M\left(\int_0^T|f|(t)|^2\,dt\right)^{p/2}$$

(see Ref. 8, p. 432) shows that one must prove the following formula (as in case of the Gaussian processes):

$$M_x|\tilde{N}_L^*(\lambda+h)-\tilde{N}_L^*(\lambda)|^2\leqslant \text{const}\cdot h^\beta,\qquad \beta>0$$

But

$$\begin{aligned}
&M_x|\tilde{N}_L^*(\lambda+h)-\tilde{N}_L^*(\lambda)|^2\\
&=\frac{1}{2L}M\left\{\int_{-L}^{L}\left[\operatorname{grad}_x u_{\lambda+h}(x_s,\theta_{\lambda+h}(s))-\operatorname{grad}_x u_\lambda(x_s,\theta_\lambda(s)),dw_s^2\right]\right\}\\
&=\frac{1}{2L}\int_{-L}^{L}M_x|\operatorname{grad}_x u_{\lambda+h}(x_s,\theta_{\lambda+h}(s))-\operatorname{grad}_x u_\lambda(x_s,\theta_\lambda(s))|^2\,ds
\end{aligned}$$

We estimate the difference between gradients under the interval. From Hörmander's theory we obtain that the function $u_\lambda(x,\theta)$ is infinitely differentiable in all three arguments: $\lambda \in \Delta$, $(x,\theta) \in S^\nu \times S^1$. Hence,

$$\begin{aligned}
&|\mathrm{grad}_x u_{\lambda+h}(x_s,\theta_{\lambda+h}(s)) - \mathrm{grad}_x u_\lambda(x_s,\theta_\lambda(s))| \\
&\quad \leqslant |\mathrm{grad}_x u_{\lambda+h}(x_s,\theta_\lambda(s)) - \mathrm{grad}_x u_\lambda(x_s,\theta_\lambda(s))| \\
&\qquad + |\mathrm{grad}_x u_{\lambda+h}(x_s,\theta_{\lambda+h}(s)) - \mathrm{grad}_x u_{\lambda+h}(x_s,\theta_\lambda(s))| \\
&\quad \leqslant C_1|h| + C_2|\theta_{\lambda+h}(s) - \theta_\lambda(s)| \\
&\quad \leqslant C_1|h| + C_3|\theta_{\lambda+h}(s) - \theta_\lambda(s)|^{1/2-\gamma}, \quad \tfrac{1}{2} > \gamma > 0
\end{aligned}$$

where the constants C_1, C_2, C_3 are independent of L. So,

$$\begin{aligned}
&\frac{1}{2L}\int_{-L}^{L} M_x|\mathrm{grad}_x u_{\lambda+h}(x_s,\theta_{\lambda+h}(s)) - \mathrm{grad}_x u_\lambda(x_s,\theta_\lambda(s))|^2\,ds \\
&\quad \leqslant \frac{1}{2L}\int_{-L}^{L} M_x\left[2C_1^2|h|^2 + 2C_3^2|\theta_{\lambda+h}(s) - \theta_\lambda(s)|^{1-2\gamma}\right] ds \\
&\quad \leqslant C_4|h|^2 + C_5\frac{1}{2L}\int_{-L}^{L} M_x|\theta_{\lambda+h}(s) - \theta_\lambda(s)|^{1-2\gamma}\,ds.
\end{aligned}$$

We may now use the important estimate (see Ref. 3), Corollary 2′):

$$M_x\left[\left|\frac{\theta_{\lambda+h}(s) - \theta_\lambda(s)}{2h}\right|^{1-2\gamma} \theta_{\lambda+h}(-L) = \theta_\lambda(-L)\right] \leqslant C(\gamma)$$

and thus finally obtain that

$$M_x|\tilde{N}_L^*(\lambda + h) - \tilde{N}_L^*(\lambda)|^2 \leqslant C_6(\gamma)|h|^{1-2\gamma} \tag{13}$$

Theorem 1 is proved.

Remark 3. The transition from $|\theta_{\lambda+h}(s) - \theta_\lambda(s)|$ to $|\theta_{\lambda+h}(s) - \theta_\lambda(s)|^{1/2-\gamma}$ was necessary because of the fact that

$$M_x\left|\frac{\theta_{\lambda+h}(s) - \theta_\lambda(s)}{2h}\right|_{s\to\infty} \to \infty$$

(see Ref. 3).

Remark 4. The inequality (13) means the continuity of the limit process $N^*(\lambda)$ and proves that this process satisfies the Hölder condition with the index $1/2 - \gamma$ for any $\gamma > 0$. As we shall see a bit later, Theorem 2 [about the local independence of the increments of the process $N^*(\lambda)$] is equivalent to the expression $M|N^*(\lambda + h) - N^*(\lambda)|^2 \sim ch$.

In other words we must "eliminate" the small constant $\gamma > 0$ from the estimate (13). Unfortunately, $c(\gamma)_{\gamma\to\infty} \to \infty$ (see Ref. 3) and the above given argumentations are not applicable in this situation. By the fact that the limit process is a Gaussian one, to prove Theorem 2 it is sufficient to show that

$$M_x|N_L^*(\lambda + h) - N_L^*(\lambda)|^2 = c(\lambda)h + \bar{\bar{o}}(h) \tag{14}$$

and

$$M_x(N_L^*(\lambda + h) - N_L^*(\lambda))(N_L^*(\lambda' + h) - N_L^*(\lambda')) = \bar{\bar{o}}(h) \tag{15}$$

where the intervals $(\lambda, \lambda + h)$ and $(\lambda', \lambda' + h)$ do not intersect. We shall also establish that $C(\lambda) = n(\lambda)$ is a state density of the operator H, i.e., $n(\lambda) = \partial N(\lambda)/\partial\lambda$.

The proof of these facts is rather complicated technically. It is essentially based on the results of Ref. 3, Section 3, describing the moments of the flow of the level intersections by a non-Gaussian random process. The estimates of the transition density of the "almost degenerate" process $(x_t, \theta_\lambda(t), \theta_{\lambda'}(t))$ with the "small" difference between λ and λ' is also an important factor in our proof. Therefore, we shall only outline the general scheme of the discussion and omit technical details.

Let us write down $M_x N_L(\Delta)$ and $M_x(N_L(\Delta)N_L(\Delta'))$ in another form. Moving the origin to $(-L)$, we can symbolically write down that

$$\begin{aligned} N_L(\Delta) &= N_L(\lambda + h) - N_L(\lambda) \\ &= \int_\Delta d\lambda \cdot \delta(\theta_\lambda(2L)) \left| \frac{\partial\theta_\lambda(2L)}{2\lambda} \right| = \int_\Delta d\lambda \cdot \delta(\theta_\lambda(2L)) \cdot Z_\lambda(2L) \end{aligned}$$

in the notations of Refs. 3 and 2.

Lemma 4. (α) For every Δ

$$\begin{aligned} M_{(x,0)}N_L(\lambda) &= \int_\Delta d\lambda \cdot M_{(x,0,0)}(Z_\lambda(2L))\delta(\theta_\lambda(2L)) \\ &= \int_\Delta d\lambda \cdot M_{(x,0,0)} \int_0^{2L} ds \\ &\quad \times \exp\left[-\int_s^{2L} \sin 2\theta_\lambda(u)(1 + F(x_u) - \lambda)\, du \right] \sin^2\theta_\lambda(s) \\ &\quad \times \delta(\theta_\lambda(2L)) = \int_\Delta d\lambda \int_0^{2L} ds \int_{S^\nu \times S^1} p_\lambda(s, (x,0), (x_1, \theta_1)) \\ &\quad \times \sin^2\theta_1 p_\lambda(2L - s, (S^\nu, 0), (x_1 - \theta_1))\, dx_1\, d\theta_1, \end{aligned} \tag{16}$$

where $p_\lambda(\cdot, (S^\nu, 0), (\cdot, \cdot))$ denotes an averaging of the density $p_\lambda(\cdot, (x, 0),$

$(\cdot,\cdot)$) with respect to x, i.e., an integration with respect to the Euclidean measure on S^ν.

$$(\beta)\quad M_x(N_L(\Delta))^2 = M_x N_L(\Delta) + \iint\limits_{\Delta\times\Delta'} d\lambda d\lambda' M_{(x,0,0)} \int_0^{2L}\int_0^{2L} ds\,ds'$$
$$\times \exp\left[\int_s^{2L} \sin 2\theta_\lambda(u)(1+F(x_u)-\lambda)\,du\right] \sin^2\theta_\lambda(s)$$
$$\times \exp\left[-\int_{s'}^{2L} \sin 2\theta_{\lambda'}(u)(1+F(x_u)-\lambda')\,du\right]$$
$$\times \sin^2\theta_{\lambda'}(s')\delta(\theta_\lambda(2L))\delta(\theta_{\lambda'}(2L))$$
$$= M_x N_L(\Delta) + 2\iint\limits_{\Delta\times\Delta'} d\lambda\,d\lambda' \iint\limits_{0<s<s'<2L} ds\,ds'$$
$$\times \int (S^\nu \times S^1 \times S^1) p_{\lambda,\lambda'}(s,(x,0,0),(x_1,\theta_1,\theta_2)) M_{(x_1,\theta_1,\theta_2)}$$
$$\times \left\{ \sin^2\theta_1 \times \exp\left[-\int_s^{s'} \sin 2\theta_\lambda(u)(1+F(x_u)-\lambda)\,du + \sin^2\theta_2\right]\right.$$
$$\left.\times \exp\left[-\int_s^{s'} \sin 2\theta_{\lambda'}(u)(1+F(x_u)-\lambda')\,du\right]\right\}$$
$$\times p_{\lambda,\lambda'}(2L-s',(S^\nu,0,0),(x_{s'-s},-\theta_\lambda(s'-s),$$
$$-\theta_{\lambda'}(s'-s)))\,dx_1\,d\theta_1\,d\theta_2 \qquad (17)$$

(γ) If the intervals Δ and Δ' do not intersect then

$$M_x N_L(\Delta) N_L(\Delta') = 2\iint\limits_{\Delta\times\Delta'} d\lambda\,d\lambda' \iint\limits_{0<s<s'<2L} ds\,ds' \qquad (18)$$

and so on as in (17).

Proof. These formulas follow from the general theory of Ref. 3, Section 3. Only one point needs additional argumentation. We shall illustrate that for (β).

It is true, that

$$M_{(x,\theta_1,\theta_2)} \exp\left[-\int_0^t \sin 2\theta_\lambda(u)(1+F(x_u)-\lambda)\,du\right.$$
$$\left.-\int_0^t \sin 2\theta_{\lambda'}(u)(1+F(x_u)-\lambda')\,du\right]$$
$$\times p_{\lambda\lambda}(t,(S^\nu,0,0),(x,-\theta_1,-\theta_2))\delta(\theta_\lambda(t))\cdot\delta(\theta_\lambda\cdot(t)) \qquad (19)$$

Indeed, $(x_t,\theta_\lambda(t),\theta_{\lambda'}(t))$ is a diffusion process with the infinitesimal operator A_2 [see (7)]. According to the Kac–Feynman formula, the expectation in (19) [let us denote it by $U(t,x,\theta_1,\theta_2)$] satisfies the parabolic differential equation

$$\frac{\partial U}{\partial t} = A_2 U - U\left[\sin 2\theta_1 \times (1 + F(x) - \lambda) + \sin 2\theta_2\left(1 + F(x) - \lambda^1\right)\right]$$

$$= \frac{1}{2}\Delta_x U + \left[\cos^2\theta_1 + (F(x) - \lambda)\sin^2\theta_1\right]$$

$$\times \frac{\partial U}{\partial \theta_1} - U\left[\sin 2\theta_1 \times (1 + F(x) - \lambda)\right]$$

$$+ \left[\cos^2\theta_2 + \left(F(x) - \lambda^1\right)\sin^2\theta_2\right]\frac{\partial U}{\partial \theta_2} - U\left[\sin 2\theta_2 \times \left(1 + F(x) - \lambda^1\right)\right]$$

$$= \frac{1}{2}\Delta_x U + \frac{\partial}{\partial \theta_1}\left[U\left(\cos^2\theta_1 + (F(x) - \lambda)\sin^2\theta_1\right)\right]$$

$$+ \frac{\partial}{\partial \theta_2}\left[U\left(\cos^2\theta_2 + \left(F(x) - \lambda^1\right)\sin^2\theta_2\right)\right]$$

with the initial condition $U(0, x, \theta_1, \theta_2) = \delta(\theta_1)\delta(\theta_2)$. Let us denote the fundamental solution of this parabolic equation by $q_{\lambda,\lambda'}(t, (x, \theta_1, \theta_2), (\tilde{x}, \tilde{\theta}_1, \tilde{\theta}_2))$, then

$$U(t, x, \theta_1, \theta_2) = \int_{S'' \times S' \times S^1} q_{\lambda,\lambda^1}(t, (x, \theta_1, \theta_2), (\tilde{x}, \tilde{\theta}_1, \tilde{\theta}_2))\delta(\theta_1)\, d\tilde{x}\, d\tilde{\theta}_1\, d\tilde{\theta}_2$$

$$= q_{\lambda,\lambda'}(t, (x, \theta_1, \theta_2), (S'', 0, 0))$$

but the function $q_{\lambda,\lambda'}$ satisfies in arguments $t, \tilde{x}, \tilde{\theta}_1, \tilde{\theta}_2$ the adjoint problem with the elliptic operator

$$\frac{1}{2}\Delta_{\tilde{x}} - \left[\cos^2\tilde{\theta}_1 + (F(\tilde{x}) - \lambda)\sin^2\tilde{\theta}_1\right]\frac{\partial U}{\partial \tilde{\theta}_1} - \left[\cos^2\tilde{\theta}_2 + (F(\tilde{x}) - \lambda')\sin^2\tilde{\theta}_2\right]\frac{\partial U}{\partial \tilde{\theta}_2}$$

on the right. Clearly, the last expression is the infinitesimal operator of the process $(x_t, -\theta_\lambda(t), -\theta_{\lambda'}(t))$, hence

$$q_{\lambda,\lambda'}(t, (x, \theta_1, \theta_2), (S'', 0, 0)) = p_{\lambda,\lambda'}(t, (S'', 0, 0), (x, -\theta_1, -\theta_2))$$

This fact proves both formula (19) and Lemma 4.

In the future we shall need the expectation of the form

$$U(t, x, \theta_1, \theta_2) = M_{(x,\theta_1,\theta_2)}\left\{\exp\left[-\int_0^t \sin 2\theta_2(u)(1 + F(x_u) - \lambda)\, du\right] \times h(x_t, \theta_\lambda(t), \theta_{\lambda'}(t))\right\}$$

Evidently, this function satisfies the parabolic equation

$$\frac{\partial U}{\partial t} = \frac{1}{2}\Delta_x U + \frac{\partial}{\partial \theta_1}\left[U\left(\cos^2\theta_1 + (F(x) - \lambda)\sin^2\theta_1\right)\right]$$

$$+ \left[\cos^2\theta_2 + (F(x) - \Lambda')\sin^2\theta_2\right]\frac{\partial U}{\partial \theta_2}, \quad U(0, x, \theta_1, \theta_2) = h(x, \theta_1, \theta_2) \quad (20)$$

If $h \equiv 1$, then

$$U(t,x,\theta_1,\theta_2) = M_{(x,\theta_1,\theta_2)}\exp\left[-\int_0^t \sin 2\theta_\lambda(U)(1+F(x_u)-\lambda)\,du\right]$$
$$= p_\lambda(t,(S^\nu,S),(x,-\theta_1)) \underset{t\to\infty}{\to} \pi_\lambda(x,-\theta_1) \tag{21}$$

with the exponential speed. Now we denote the fundamental solution of (20) by $\tilde{p}^{(\lambda)}_{\lambda,\lambda'}(t,(x,\theta_1,\theta_2),(\tilde{x},\tilde{\theta}_1,\tilde{\theta}_2))$ and the corresponding semigroup of the operators on $C(S^\nu \times S^1 \times S^1)$ by $\tilde{P}_t$. Since this semigroup is evidently positive we have that the "largest" eigenvalue of the infinitesimal operator of the semigroup equals zero [by the Frobenius theorem together with (21)]. We obtain also that $\pi_\lambda(x,-\theta_1)$ is the eigenfunction corresponding to the "largest" eigenvalue, $\pi_{\lambda'}(x,\theta_2)$ is the "largest" eigenfunction of the adjoint problem, and

$$\tilde{p}^{(\lambda)}_{\lambda,\lambda'}(t,(x,\theta_1,\theta_2),(\tilde{x},\tilde{\theta}_1,\tilde{\theta}_2)) \underset{t\to\infty}{\to} \pi_\lambda(x,-\theta_1)\pi_{\lambda'}(\tilde{x},\tilde{\theta}_2) \tag{22}$$

with the exponential speed.

We can represent $M_x(N_L(\Delta))^2$ in terms of the fundamental solutions described above. Thus we have

$$\begin{aligned}
M_x(N_L(\Delta))^2 = {} & \int_\Delta d\lambda \int_0^{2L} ds \int_{S^\nu\times S^1} p_\lambda(s,(x,0),(x_1,\theta_1))\sin^2\theta_1 \\
& \times p_\lambda(2L-s,(S^\nu,0),(x_1,-\theta_1))\,dx_1\,d\theta_1 \\
& + 2\iint_{\Delta\times\Delta'} d\lambda\,d\lambda' \iint_{0<s<s'<2L} ds\,ds' \\
& \times \iint_{(S^\nu\times S^1\times S'^1)^2} p_{\lambda,\lambda'}(s,(x,0,0),(x_1,\theta_1,\theta_2)) \\
& \times \Big[\sin^2\theta_1\sin^2\tilde{\theta}_2\,\tilde{p}^{(\lambda)}_{\lambda,\lambda'}(s'-s,(x_1,\theta_1,\theta_2), \\
& \times(\tilde{x},\tilde{\theta}_1,\tilde{\theta}_2)) + \sin^2\theta_2\sin^2\tilde{\theta}_1\,\tilde{p}^{(\lambda')}_{\lambda,\lambda'}(s'-s,(x_1,\theta_1,\theta_2), \\
& \times(\tilde{x},\tilde{\theta}_1,\tilde{\theta}_2))\Big]\,p_{\lambda,\lambda'}(2L-s',(S^\nu,0,0),(\tilde{x},-\tilde{\theta}_1,-\tilde{\theta}_2)) \\
& \times dx_1\,d\theta_1\,d\theta_2\,d\tilde{x}\,d\tilde{\theta}_1\,d\tilde{\theta}_2
\end{aligned} \tag{23}$$

The first term of (23) is equivalent to

$$2L\int_\Delta d\lambda \int_{S^\nu\times S^1} \pi_\lambda(x_1,\theta_1)\sin^2\theta_1\,\pi_\lambda(x_1,-\theta_1)\,dx_1\,d\theta_1 = 2L\int_\Delta n(\lambda)\,d\lambda$$

for the large L and $n(\lambda)$ is the state density of the initial operator H, $n(\lambda) > 0$ for all $\lambda > 0$.

We may add the limits of the fundamental solutions $\tilde{p}^{(\lambda)}_{\lambda,\lambda'}$ and $\tilde{p}^{(\lambda')}_{\lambda,\lambda'}$ to the expression in square brackets under the second integral and then subtract these limits, as is usually done when counting the variance of the additive functional from the Markov process. In the end we obtain two integrals:

$$I_1 = 2\iint_{\Delta\times\Delta'} d\lambda\, d\lambda' \iint_{(S^\nu\times S^1\times S^1)^2} ds\, ds' \iint_{0<s<s'<2L} p_{\lambda,\lambda'}(s,(x,0,0),(x_1,\theta_1,\theta_2))$$
$$\times\left[\sin^2\theta_1\times\pi_\lambda(x_1,-\theta_1)\pi_{\lambda'}(\tilde{x},\tilde{\theta}_2)+\sin^2\theta_2\times\pi_\lambda(\tilde{x},-\tilde{\theta}_2)\pi_{\lambda'}(x_1,\theta_1)\right]$$
$$\times p_{\lambda,\lambda'}(2L-s',(S^\nu,0,0),(\tilde{x},-\tilde{\theta}_1,-\tilde{\theta}_2))\,dx_1\,d\theta_1\,d\theta_2\,d\tilde{x}\,d\tilde{\theta}_1\,d\tilde{\theta}_2$$
$$I_2 = 2\iint_{\Delta\times\Delta'} d\lambda\, d\lambda' \cdots \left[\sin^2\theta_1\left(\tilde{p}^{(\lambda)}_{\lambda,\lambda'}-\pi_\lambda\pi_{\lambda'}\right)\right.$$
$$\left.+\sin^2\theta_2\left(\tilde{p}^{(\lambda')}_{\lambda,\lambda'}-\pi_\lambda\pi_{\lambda'}\right)\right]\cdots$$

It is easy to see that $I_1 = M^2N_L(\Delta)+O(e^{-\delta L})\sim 4L^2\int_{\Delta\times\Delta'}\int n(\lambda)n\cdot(\lambda')\,d\lambda\,d\lambda'$. This equality is a consequence of both the properties of the invariant measure and of that fact, that the arguments under integral are as a matter of fact separable.

As for integral I_2, it has the order L. The standard argumentations show that

$$\lim_{L\to\infty}\frac{I_2}{2L} = 2\iint_{\Delta\times\Delta'} d\lambda\, d\lambda' \int_0^\infty d\tau \iint_{(S^\nu\times S^1\times S^1)^2} \pi_{\lambda,\lambda'}(x_1,\theta_1,\theta_2)$$
$$\times\left[\sin^2\theta_1\left(\tilde{p}^{(\lambda)}_{\lambda,\lambda'}(\tau,(x_1,\theta_1,\theta_2),(\tilde{x},\tilde{\theta}_1,\tilde{\theta}_2))-\pi_\lambda(x_1,-\theta_1)\pi_{\lambda'}(\tilde{x},\tilde{\theta}_2)\right)\right.$$
$$\left.+\sin^2\theta_2\left(\tilde{p}^{(\lambda')}_{\lambda,\lambda'}(\tau,(x_1,\theta_1,\theta_2),(\tilde{x},\tilde{\theta}_1,\tilde{\theta}_2))-\pi_\lambda(\tilde{x},-\tilde{\theta}_2)\pi_{\lambda'}(x_1,\theta_1)\right)\right]$$
$$\times\pi_{\lambda,\lambda'}(\tilde{x},-\tilde{\theta}_1,-\tilde{\theta}_2)\,dx_1\,d\theta_1\,d\theta_2\,d\tilde{x}\,d\tilde{\theta}_1\,d\tilde{\theta}_2$$

This limit transition is trivial for λ,λ' such that $|\lambda'-\lambda|>\epsilon$. For λ' close to λ it is necessary to apply the estimates of this density established in Ref. 3 because of the degeneration of the transition density $p_{\lambda,\lambda'}$.

Thus we have definitely proved that

$$\lim_{L\to\infty} D_x\left(\frac{N_L(\Delta)}{(2L)^{1/2}}\right)^2 = \int_\Delta n(\lambda)\,d\lambda + \iint_{\Delta\times\Delta'} K(\lambda,\lambda')\,d\lambda\,d\lambda',$$

where $K(\lambda,\lambda')$ is some integrable continuous for $\lambda=\lambda'$ function. If $\Delta=[\lambda,\lambda+h]$, then $\int_\Delta n(\lambda)\,d\lambda = |h|n(\lambda)+\bar{\bar{o}}(h)$ and $\int_{\Delta\times\Delta'}|K(\lambda,\lambda')|\,d\lambda\,d\lambda' = \bar{\bar{o}}(h)$, i.e.,

$$M(N^*(\lambda+h)-N^*(\lambda))^2 = n(\lambda)h+\bar{\bar{o}}(h)$$

As far as a covariance is concerned, then

$$\lim_{L\to\infty} \frac{\operatorname{cov}(N_L(\Delta), N_L(\Delta'))}{2L} = \int\int_{\Delta\times\Delta'} K(\lambda,\lambda')\,d\lambda\,d\lambda', \Delta\cap\Delta' = \emptyset,$$

whence $M(N^*(\lambda + h) - N^*(\lambda))(N^*(\lambda' + h) - N^*(\lambda')) = \bar{\bar{o}}(h)$. Theorem 2 is proved. ■

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